DISCRETE EVENT DYNAMIC SYSTEMS

MARKOV CHAINS

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MARKOV CHAINS
Outline

- Definition of Markov Chain (MC) and relation to DES.

- Discrete Time MCs
  - the transition probability matrix
  - homogeneous MCs
  - state holding times
  - state probabilities
  - transient analysis
  - classification of states
  - steady-state analysis

- Continuous Time MCs
  - the transition rate matrix
  - homogeneous MCs
  - transition probabilities
  - state probabilities
  - transient analysis
  - steady-state analysis

MARKOV CHAINS
Definitions

Def.: A Markov Chain is a discrete state space stochastic process where the probability of transitions between states has the following property:

$$P[X(t_{k+1}) = x_{k+1} \mid X(t_k) = x_k, X(t_{k-1}) = x_{k-1}, \cdots] = P[X(t_{k+1}) = x_{k+1} \mid X(t_k) = x_k]$$

Recall that, in a Markov process:

- All past state information is irrelevant (no *state memory* needed).
- How long the process has been in the current state is irrelevant (no state *age memory* needed).

Discrete Time Markov Chains (DTMC)

Stochastic sequence $\{X_1, X_2, \ldots\}$ characterized by the Markov property:

$$P[X_{k+1} = x_{k+1} \mid X_k = x_k, X_{k-1} = x_{k-1}, \cdots] = P[X_{k+1} = x_{k+1} \mid X_k = x_k]$$

RELATION WITH DES

Relation with STA: We will only be concerned with the total probability of making a transition from state *x* to state *x'*, regardless of which event causes the transition:

$$p(x' \mid x) = P[X(t_{k+1}) = x' \mid X(t_k) = x] = \sum_{i \in \Gamma(x)} p(x' \mid x, i).p(i \mid x)$$

Therefore, to specify a (CT)MC model, we will only need to identify:

1. A state space χ
2. An initial state probability $p_0(x)=P[X_0=x]$, for all $x \in \chi$
3. Transition probabilities $p(x',x)$

Relation with ETPN: The marking process of an exponential timed Petri net is a continuous time Markov Chain (CTMC).

DISCRETE TIME MARKOV CHAINS (DTMC)

Transition probabilities

$$p_{ij}(k) \equiv P[X_{k+1} = j \mid X_k = i]$$

$$0 \leq p_{ij}(k) \leq 1$$

$$\sum_{\text{all } j} p_{ij}(k) = 1$$

***n*-step transition probabilities**

$$p_{ij}(k, k+n) \equiv P[X_{k+n} = j \mid X_k = i]$$

$$p_{ij}(k, k+n) = \sum_{\text{all } r} P[X_{k+n} = j \mid X_u = r, X_k = i].P[X_u = r \mid X_k = i], \; k < u \leq k+n$$

Chapman-Kolmogorov Equations

$$p_{ij}(k, k+n) = \sum_{\text{all } r} p_{ir}(k, u) p_{rj}(u, k+n), \; k < u \leq k+n$$

DISCRETE TIME MARKOV CHAINS (DTMC)

Chapman-Kolmogorov Equations (Matrix Form)

$$\mathbf{H}(k,k+n) \equiv [p_{ij}(k,k+n)], \quad i,j = 0,1,2,\ldots$$

$$p_{ij}(k,k+n) = \sum_{\text{all } r} p_{ir}(k,u)\,p_{rj}(u,k+n), \; k < u \le k+n \longrightarrow \mathbf{H}(k,k+n) = \mathbf{H}(k,u)\mathbf{H}(u,k+n)$$

Forward Chapman-Kolmogorov Equation

$$u = k+n-1 \quad \rightarrow \quad \mathbf{H}(k,k+n) = \mathbf{H}(k,k+n-1)\mathbf{H}(k+n-1,k+n)$$

Backward Chapman-Kolmogorov Equation

$$u = k+1 \quad \rightarrow \quad \mathbf{H}(k,k+n) = \mathbf{H}(k,k+1)\mathbf{H}(k+1,k+n)$$

HOMOGENEOUS DTMC

Homogeneous MCs $P[X(k+1)=j \mid X(k)=i] = \text{constant} = p_{ij}$

The transition probabilities are independent of time *k*. Note that not all probabilities involved (e.g., joint probabilities) are time-independent.

$$p_{ij}^n \equiv P[X_{k+n}=j \mid X_k = i], \quad n = 1,2,\ldots$$

$$\mathbf{H}(k,k+n) = \mathbf{H}(n) \equiv [p_{ij}^n], \quad i,j = 0,1,2,\ldots$$

Setting $u = k+m$ in the CK equation:

$$p_{ij}^n = \sum_{\text{all } r} p_{ir}^m p_{rj}^{n-m} \overset{m=n-1}{=} \sum_{\text{all } r} p_{ir}^{n-1} p_{rj} \xrightarrow{m=n-1}$$

CK equation

$$\mathbf{H}(n) = \mathbf{H}(n-1)\mathbf{H}(1)$$

HOMOGENEOUS DTMC

Transition Probability Matrix

$$\mathbf{P} \equiv [p_{ij}] = \mathbf{H}(1), \quad i, j = 0,1,2,...$$

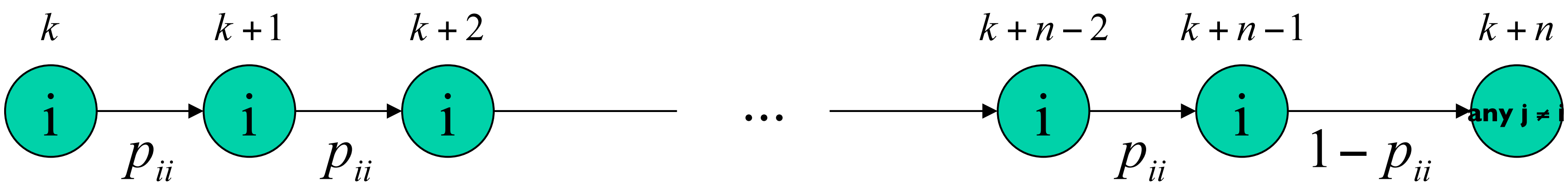
$$p_{ij}^{n} = \sum_{\text{all } r} p_{ir}^{m} p_{rj}^{n-m} = \sum_{\text{all } r} p_{ir}^{n-1} p_{rj}$$

CK equation

$$\mathbf{H}(n) = \mathbf{H}(n-1)\mathbf{P}$$

State Holding Times (Sojourn Times)

V(i) Random variable representing the number of consecutive time steps spent at state *i*



$$P[V(i)=n] = P[X_{k+1}=i, X_{k+2}=i, \cdots X_{k+n-1}=i, X_{k+n}\neq i \mid X_k = i]$$

$$P[V(i)=n] = (1-p_{ii})\,p_{ii}^{n-1}$$

Geometric distribution with parameter p_{ii}

HOMOGENEOUS DTMC

State Probabilities

$$\pi_j(k) \equiv P[X_k = j]$$

$$\pi(k) = [\pi_0(k), \pi_1(k), \ldots]$$

$$0 \le \pi_j(k) \le 1$$

$$\sum_{\text{all } j} \pi_j(k) = 1$$

If, in addition to the state space χ and the transition probability matrix **P** the initial state probability vector $\pi(0) = [\pi_0(0), \pi_1(0), \ldots]$ is specified, the DTMC is completely specified.

Two types of analysis will be carried out:

- transient analysis
- steady-state analysis

HOMOGENEOUS DTMC
State Probabilities

Transient Analysis

$$\pi_j(k+1) = P[X_{k+1} = j] = \sum_{\text{all } i} P[X_{k+1} = j \mid X_k = i] P[X_k = i] = \sum_{\text{all } i} p_{ij}\pi_i(k)$$

$$\pi(k+1) = \pi(k)\mathbf{P}, \quad k = 0,1,\ldots$$

Solution:

$$\pi(k) = \pi(0)\mathbf{P}^k, \quad k = 1,2,\ldots$$

HOMOGENEOUS DTMC
State Classification

Reachable

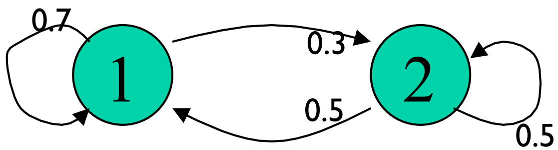
j is reachable from *i* if there is a path from *i* to *j*, i.e., if $p_{ij}^n > 0$ for some *n*=1,2,…

Absorbing

A subset *S* of the state space χ is said to be **closed** if $p_{ij}=0$ for any $i \in S$, $j \notin S$.
State *i* is **absorbing** if it forms a single-element closed set ($p_{ii}=1$).

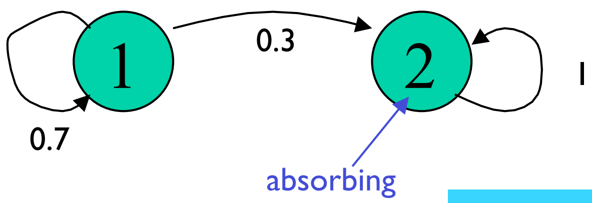
$$i \text{ is absorbing} \quad \Leftrightarrow \quad \exists_{k_0}, \forall_{k \geq k_0} : \pi_i(k) = 1$$

Irreducible



A closed set of states *S* is irreducible if state *j* is reachable from state *i* for any $i,j \in S$.
A MC is **irreducible** if its state space χ is irreducible.

Reducible, when there are subsets of the state space not reachable from other states (e.g., state 1 from 2 in the MC on the right)



HOMOGENEOUS DTMC
State Classification

Q.: The MC is in state *i.* Will the chain ever return to state *i*?

A.:

- **definitely yes:** state *i* is **recurrent**
- **maybe no:** state *i* is **transient**

first time the chain enters *j*, starting in *i*

Hitting time: $T_{ij} \equiv \min\{k > 0 : X_0 = i, X_k = j\}$

Recurrence time: $T_{ii} \equiv \min\{k > 0 : X_0 = i, X_k = i\},\quad T_{ii} = 1,2,\ldots,\infty$

$$\rho_i^k \equiv P[T_{ii} = k]$$

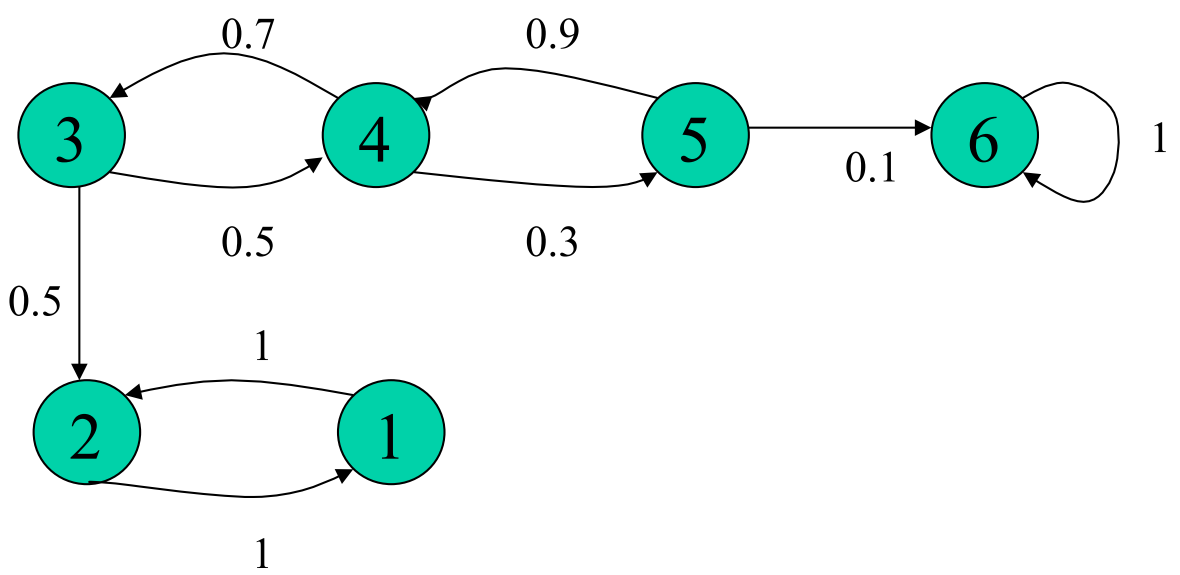
$\rho_i \equiv \sum_{k=1}^{\infty} \rho_i^k$ = P [ever return to *i* | current state is *i*] = $P\,[T_{ii}<\infty\,]$

HOMOGENEOUS DTMC
State Classification

Recurrent state i is recurrent if $\rho_i = 1$

Transient state i is transient if $\rho_i < 1$

Example



1,2,6 - recurrent

6 - absorbing

3,4,5 - transient

1,2 is reachable from 1,2,3,4,5 ;
3,4,5 from 3,4,5; 6 from 3,4,5,6

{1,2}, {6} – closed sets

HOMOGENEOUS DTMC
State Classification

Theorem 1: If a MC has a finite state space, then at least some state is *recurrent*.

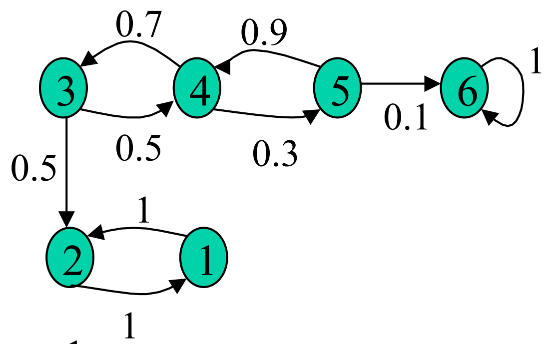
Theorem 2: If i is a *recurrent* state and j is reachable from i, then state j is *recurrent*.

Theorem 3: If S is a *finite closed irreducible* set of states, then every state in S is *recurrent*.

HOMOGENEOUS DTMC
State Classification

The *mean recurrence time* is $M_i \equiv E[t_{ii}] = \sum_{k=1}^{\infty} k\rho_i^k$

Null recurrent	If the mean recurrence time is $M_i = \infty$
Positive recurrent	If the mean recurrence time is $M_i < \infty$



Ex.: positive recurrent states

State 1 $\quad E[t_{11}] = \sum_{k=1}^{\infty} k\rho_1^k = 1\cdot 0 + 2\cdot 1 + 3\cdot 0 + \cdots = 2 \quad \rho_1 = 1$

State 2 $\quad E[t_{22}] = \sum_{k=1}^{\infty} k\rho_2^k = 1\cdot 0 + 2\cdot 1 + 3\cdot 0 + \cdots = 2 \quad \rho_2 = 1$

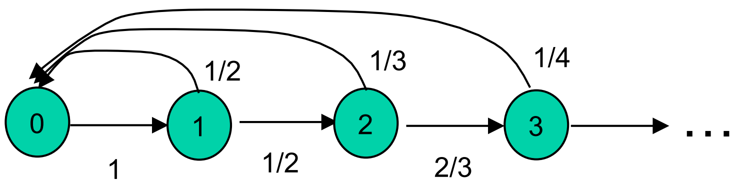
State 6 $\quad E[t_{66}] = \sum_{k=1}^{\infty} k\rho_6^k = 1\cdot 1 + 2\cdot 0 + 3\cdot 0 + \cdots = 1 \quad \rho_6 = 1$

HOMOGENEOUS DTMC
State Classification

The *mean recurrence time* is $M_i \equiv E[t_{ii}] = \sum_{k=1}^{\infty} k \rho_i^k$

Null recurrent	If the mean recurrence time is $M_i = \infty$
Positive recurrent	If the mean recurrence time is $M_i < \infty$

Ex.: null recurrent states



State 0 is

$$\rho_0 \equiv \sum_{k=1}^{\infty} \rho_0^k = \sum_{k=2}^{\infty} \frac{1}{k-1}\frac{1}{k} = \sum_{k=1}^{\infty} \frac{1}{k}\frac{1}{k+1} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \lim_{n \to \infty} \left(1 - \frac{1}{n}\right) = 1$$

recurrent

$$M_0 \equiv \sum_{k=1}^{\infty} k \rho_0^k = \sum_{k=2}^{\infty} \frac{1}{k-1} = \sum_{k=1}^{\infty} \frac{1}{k} = \infty$$

null recurrent

HOMOGENEOUS DTMC
State Classification

Transient states may never be revisited

Positive recurrent will definitely be revisited with finite expected recurrence time

Null recurrent will definitely be revisited but the expected recurrence time is infinite

HOMOGENEOUS DTMC
State Classification

Theorem 4: If *i* is a *positive recurrent* state and *j* is reachable from *i*, then state *j* is *positive recurrent*.

Theorem 5: If *S* is a *closed irreducible* set of states, then every state in *S* is *positive recurrent* or every state in *S* is *null recurrent* or every state in *S* is *transient*.

Theorem 6: If *S* is a *finite closed irreducible* set of states, then every state in *S* is *positive recurrent*.

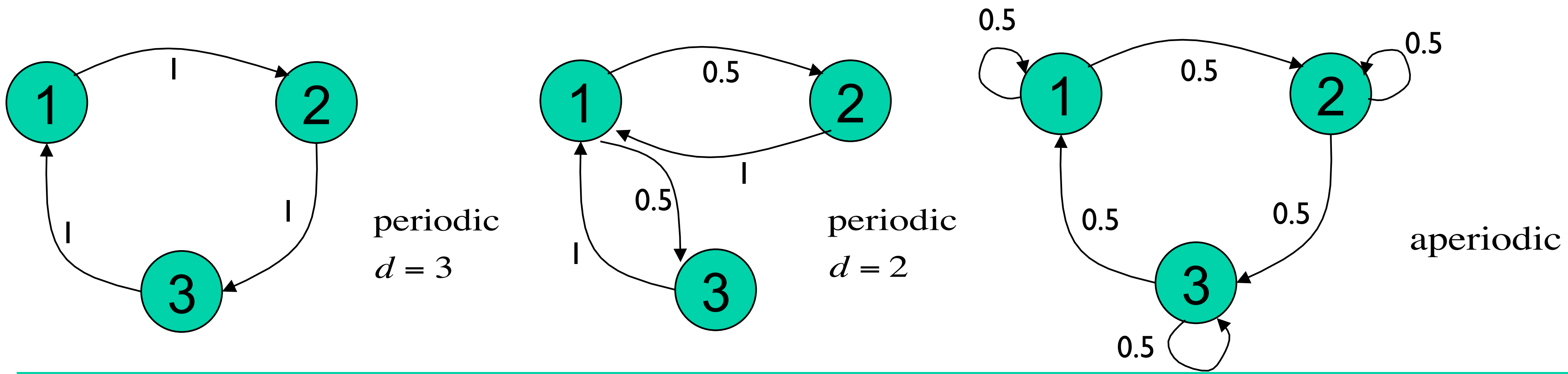
HOMOGENEOUS DTMC
State Classification

Def.: A state i is said to be *periodic* if the greatest common divisor d of the set $\{n>0: p_{ii}^n>0\}$ is $d \geq 2$. If $d=1$, the state is said to be *aperiodic*.

Periodic — State is visited every d steps

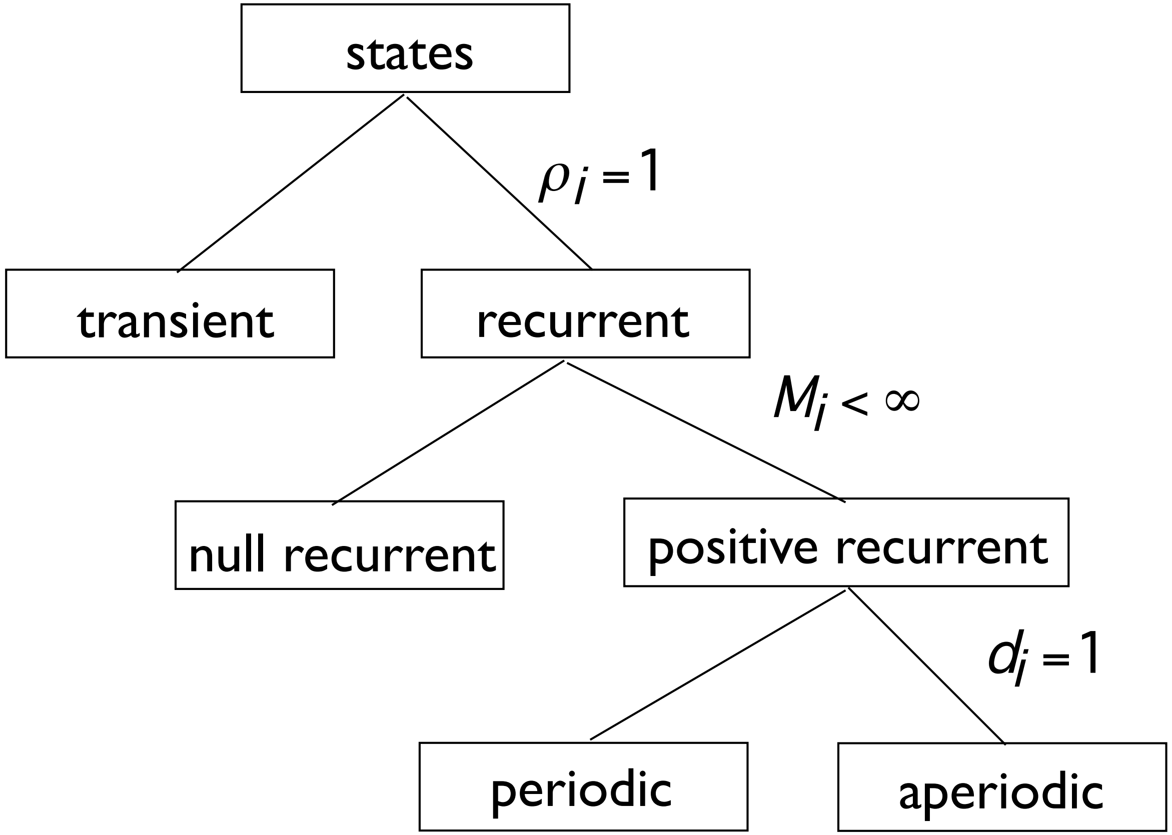
Aperiodic — There's no d such that the state is visited regularly every d steps

Examples:



Theorem 7: If a MC is irreducible, then all its states have the same period.

HOMOGENEOUS DTMC
Summary of State Classification



HOMOGENEOUS DTMC
State Probabilities

Steady-State Analysis

Q.: What is the probability of finding a MC at state *i* in the long run, i.e., after a period of time long enough so that the state probabilities have reached fixed values which do not change with time?

$$\pi_j = \lim_{k \to \infty} \pi_j(k)$$

Issues to be addressed:

- under what conditions do the above limits exist?
- if they exist, do they form a probability distribution, i.e., $\sum_{\text{all } j} \pi_j = 1$?
- how do we evaluate π_j ?

If π_j exists for some state *j*, it is referred as the *steady-state*, *equilibrium* or *stationary state probability.* If this is true for all states *j*, we obtain the *stationary probability vector* $\pi = [\pi_0, \pi_1, \ldots]$

HOMOGENEOUS DTMC
State Probabilities

Steady-State Analysis

If the limits exist $\pi_j(k+1) = \pi_j(k) \Rightarrow \pi = \pi P$

When the MC is *periodic*, the limits do **not** exist.

On the other hand,

Theorem 8 – In an irreducible aperiodic MC the limits

$$\pi_j = \lim_{k \to \infty} \pi_j(k)$$

always exist and are independent of the initial state probability vector.

Steady-State Analysis – Irreducible MCs

Recalling Theorem 5

If S is a *closed irreducible* set of states, then every state in S is *positive recurrent* or every state in S is *null recurrent* or every state in S is *transient*.

We get to the following two fundamental Theorems:

Theorem 9: In an irreducible aperiodic MC consisting of *null recurrent* or of *transient* states

$$\pi_j = \lim_{k \to \infty} \pi_j(k) = 0$$

For all states j, and no stationary probability distribution exists.

Theorem 10: In an irreducible aperiodic MC consisting of *positive recurrent* states, a unique stationary state probability vector π exists such that $\pi_j > 0$ and

$$\pi_j = \lim_{k \to \infty} \pi_j(k) = \frac{1}{M_j}$$

HOMOGENEOUS DTMC
State Probabilities

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In Theorem 9, M_j is the *mean recurrence time*

$$M_j \equiv E[t_{jj}] = \sum_{k=1}^{\infty} k\rho_j^k$$

and the steady state probabilities are determined by solving

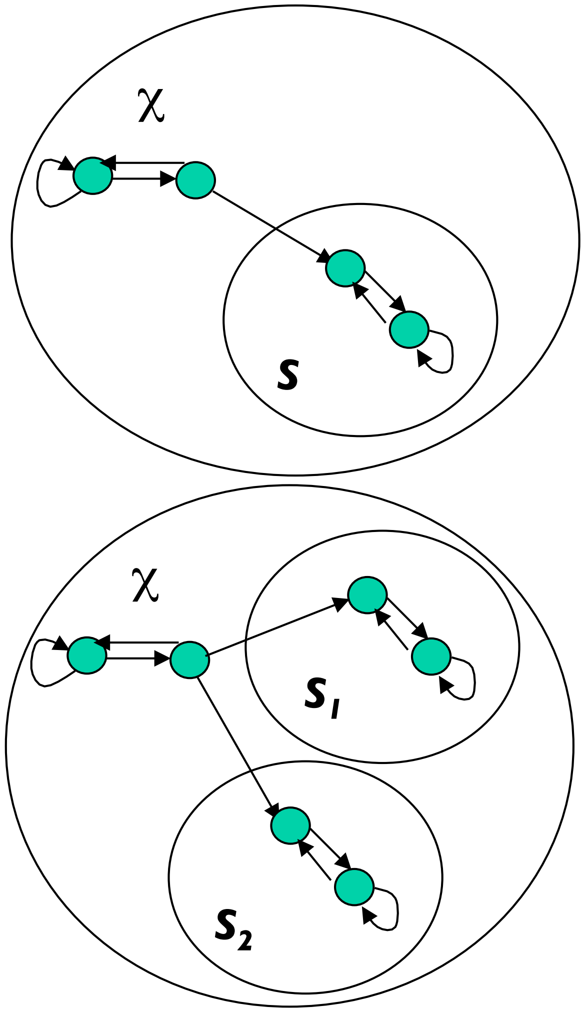
$$\pi = \pi \mathbf{P}$$
$$\sum_{\text{all } j} \pi_j = 1$$

Aperiodic positive recurrent states are very important and desirable – they are called **ergodic**. If all the states of a MC are ergodic, the MC is said to be **ergodic**.

From Theorems 6 and 10, every *finite* irreducible aperiodic MC has a unique stationary state probability vector determined by solving the above *finite* system equations. Note that solving an infinite system of equations is not so simple, though.

HOMOGENEOUS DTMC
State Probabilities

Steady-State Analysis – Reducible MCs



The chain eventually enters some irreducible closed set of states *S* and remains there forever:

• if *S* consists of 2 or more states, the steady state behavior of *S* can be analyzed as in the irreducible MC case

• if *S* consists of a single absorbing state, the MC simply remains in that state

The problem arises when the reducible chain contains two or more irreducible closed sets of states

HOMOGENEOUS DTMC
State Probabilities

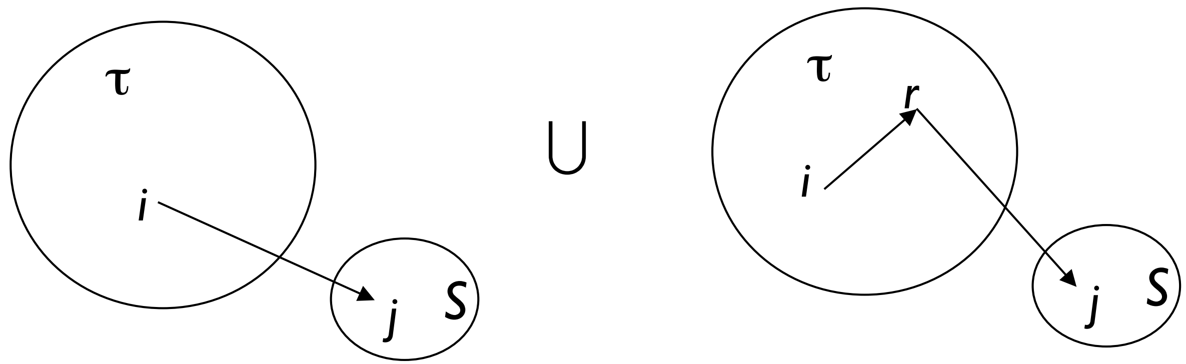
Steady-State Analysis – Reducible MCs

In this case, the relevant question is: *what is the probability that the chain enters a particular set S first*?

Def.: probability that the chain enters set *S* given that it starts at state $i \in \tau$:

$$\rho_i(S) \equiv P[X_k \in S \text{ for some } k > 0 \mid X_0 = i]$$

τ is the set of transient states in a reducible MC



$$\rho_i(S) = \sum_{j \in S} p_{ij} + \sum_{r \in \tau} \rho_r(S) p_{ir}$$

The solution for the unknown probabilities $\rho_i(s)$ for all $i \in \tau$ is not easy, but it has a unique solution for a finite set τ. However, if the set is infinite, the solution may not be unique.

CONTINUOUS TIME MARKOV CHAINS (CTMC)

The Markov (memoryless) property is expressed here as

$$P[X(t_{k+1}) = x_{k+1} \mid X(t_k) = x_k, X(t_{k-1}) = x_{k-1}, \cdots] = P[X(t_{k+1}) = x_{k+1} \mid X(t_k) = x_k],$$

$$t_0 \leq t_1 \leq \ldots \leq t_k \leq t_{k+1}$$

The analysis of CTMC parallels that of DTMC. However, the one-step probability matrix **P** can no longer be used since state transitions are no longer synchronized by a common clock.

CONTINUOUS TIME MARKOV CHAINS (CTMC)

Transition functions

$$p_{ij}(s,t) \equiv P[X(t) = j \mid X(s) = i], \; s \le t$$

$$p_{ij}(s,t) = \sum_{\text{all } r} P[X(t) = j \mid X(u) = r, X(s) = i].P[X(u) = r \mid X(s) = i]$$

Chapman-Kolmogorov Equations

$$p_{ij}(s,t) = \sum_{\text{all } r} p_{ir}(s,u)\, p_{rj}(u,t), \; s \le u \le t$$

CONTINUOUS TIME MARKOV CHAINS (CTMC)

Chapman-Kolmogorov Equations (Matrix Form)

$$\mathbf{H}(s,t) \equiv [p_{ij}(s,t)], \quad i,j = 0,1,2,\ldots$$

$$\mathbf{H}(s,s) = \mathbf{I}$$

$$\mathbf{H}(s,t) = \mathbf{H}(s,u)\mathbf{H}(u,t), \quad \mathrm{s} \le \mathrm{u} \le \mathrm{t}$$

The Transition Rate Matrix

$$\mathbf{H}(s,t+\Delta t) = \mathbf{H}(s,t)\mathbf{H}(t,t+\Delta t), \quad \mathrm{s} \le \mathrm{t} \le \mathrm{t} + \Delta t$$

$$\lim_{\Delta t \to 0} \frac{\mathbf{H}(s,t+\Delta t) - \mathbf{H}(s,t)}{\Delta t} = \mathbf{H}(s,t) \underbrace{\lim_{\Delta t \to 0} \frac{\mathbf{H}(t,t+\Delta t) - \mathbf{I}}{\Delta t}}_{\mathbf{Q}(t)} \Rightarrow \frac{\partial \mathbf{H}(s,t)}{\partial t} = \mathbf{H}(s,t)\mathbf{Q}(t), \quad s \le t$$

Transition rate matrix

CONTINUOUS TIME MARKOV CHAINS (CTMC)

Backward Chapman-Kolmogorov Equation

$$\frac{\partial \mathbf{H}(s,t)}{\partial s} = -\mathbf{Q}(s)\mathbf{H}(s,t), \quad s \le s + \Delta s \le t$$

Forward Chapman-Kolmogorov Equation

$$\frac{\partial \mathbf{H}(s,t)}{\partial t} = \mathbf{H}(s,t)\mathbf{Q}(t), \quad s \le t \le t + \Delta t$$

Solution of the FCK: (under certain conditions that **Q** must satisfy)

$$\mathbf{H}(s,t) = \exp\left[\int_s^t \mathbf{Q}(\tau)d\tau\right]$$

HOMOGENEOUS CTMC

$$p_{ij}(s,s+\tau) \equiv P[X(s+\tau)=j \mid X(s)=i] = p_{ij}(\tau)$$

$$\mathbf{H}(\tau) \rightarrow \mathbf{P}(\tau) \equiv [p_{ij}(\tau)], \quad i,j = 0,1,2,\ldots$$

$$\sum_{\text{all } j} p_{ij}(\tau) = 1$$

Note that, for a homogeneous CTMC: **H**(t,t+Δt) = **P**(Δt),

therefore **Q**(t) = **Q** = constant

$$\frac{d\mathbf{P}(\tau)}{d\tau} = \mathbf{P}(\tau)\mathbf{Q}$$

$$\text{with i.c. } p_{ij} = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases} \qquad (*)$$

Solution:

$$\mathbf{P}(t) = \exp[\mathbf{Q}t] = \mathbf{I} + \mathbf{Q}\mathrm{t} + \mathbf{Q}^2 t^2 / 2! + \ldots$$

HOMOGENEOUS CTMC

State Holding Times (Sojourn Times)

$V(i)$ Random variable representing the amount of time spent at state *i* whenever it is visited

$$P[V(i) \leq t] = 1 - e^{-\Lambda(i)t}, \qquad t \geq 0$$

Exponential distribution with parameter $\Lambda(i)$

For MC, an *event* coincides with a *state transition*, therefore “interevent times” are identical to “state holding times”.

Defining events e_{ij} *as events* generated by a Poisson process with rate λ_{ij} which cause transition from state *i* to state *j*:

$$\Lambda(i) = \sum_{e_{ij} \in \Gamma(i)} \lambda_{ij}$$

HOMOGENEOUS CTMC

Physical Interpretation of the Properties of the Transition Rate Matrix

$$\frac{dP(\tau)}{d\tau} = P(\tau)Q \quad \text{with i.c. } p_{ij} = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases} \quad \Rightarrow \quad \frac{dp_{ij}(\tau)}{d\tau} = p_{ij}(\tau)q_{jj} + \sum_{r \neq j} p_{ir}(\tau)q_{rj}$$

$$q_{ii} = \frac{d}{d\tau}\left[p_{ii}(\tau)\right]\Big|_{\tau=0}$$

Note that: $-q_{ii} = \frac{d}{d\tau}\left[1 - p_{ii}(\tau)\right]\Big|_{\tau=0} = \Lambda(i)$

$-q_{ii}$ is the *instantaneous rate* at which a state transition out of i takes place.

$$q_{ij} = \frac{dp_{ij}(\tau)}{d\tau}\Big|_{\tau=0} = \lambda_{ij}$$

q_{ij} is the *instantaneous rate* at which a state transition from i to j takes place.

$$\sum_{\text{all } j} p_{ij}(\tau) = 1 \longrightarrow \sum_{\text{all } j} q_{ij} = 0$$

Differentiating w.r.t. τ and setting $\tau=0$

HOMOGENEOUS CTMC

State following transition at the random time instant T_{k+1}

Transition Probabilities

$$P_{ij} \equiv P[X_{k+1} = j \mid X_k = i] = \frac{\lambda_{ij}}{\Lambda(i)} = \frac{q_{ij}}{-q_{ii}} \quad i \neq j$$

$$\sum_{\text{all } j \neq i} P_{ij} = 1 \Rightarrow P_{ii} = 0 \text{ (the only defined events for a MC are those causing state transitions)}$$

Once **Q** is specified, a full MC model specification is obtained:

- P_{ij} determined as above
- the parameters of the exponential state holding time are given by

$$-q_{ii} = \sum_{j \neq i} q_{ij}$$

HOMOGENEOUS CTMC

State Probabilities

$$\pi_j(t) \equiv P[X(t) = j]$$

$$\pi(t) = [\pi_0(t), \pi_1(t), \ldots]$$

$$0 \leq \pi_j(t) \leq 1$$

$$\sum_{\text{all } j} \pi_j(t) = 1$$

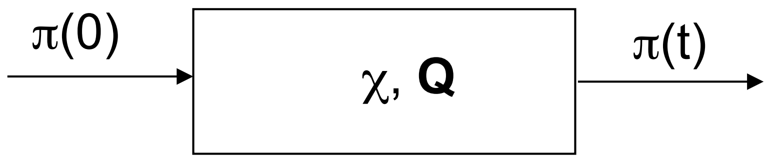
If, in addition to the state space χ and the transition matrix $\mathbf{P}(\tau)$, the initial state probability vector $\pi(0) = [\pi_0(0), \pi_1(0), \ldots]$ is specified, the CTMC is completely specified.

Notice that $\mathbf{P}(\tau) = e^{\mathbf{Q}\tau}$, therefore the specification of $\mathbf{Q}$ is enough.

Two types of analysis will be carried out:

- transient analysis
- steady-state analysis

HOMOGENEOUS CTMC
State Probabilities



Transient Analysis

$$\pi_j(t) = P[X(t) = j] = \sum_{\text{all } i} P[X(t) = j \mid X(0) = i] P[X(0) = i] = \sum_{\text{all } i} p_{ij}(t)\pi_i(0)$$

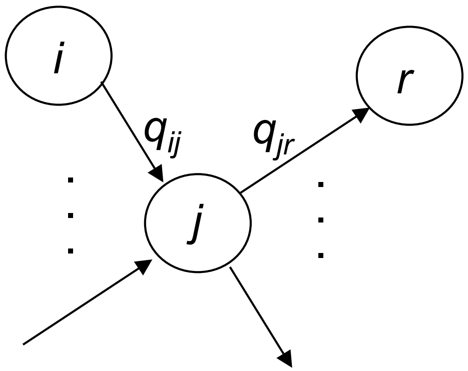
$$\pi(t) = \pi(0)\mathbf{P} = \pi(0)e^{\mathbf{Q}t}$$

This is the solution of:

$$\frac{d\pi(t)}{dt} = \pi(t)\mathbf{Q} \quad (**)$$

HOMOGENEOUS CTMC
State Probabilities

State transition rate diagram



Total flow into state $j = \sum_{i \neq j} q_{ij}\pi_i(t)$

Total flow out of state $j = \sum_{r \neq j} q_{jr}\pi_j(t)$

Net probability flow rate into state j :

$$\frac{d\pi_j(t)}{dt} = \sum_{i \neq j} q_{ij}\pi_i(t) - \left(\sum_{r \neq j} q_{jr}\right)\pi_j(t)$$

but $\sum_{r \neq j} q_{jr} = -q_{jj}$

therefore

$$\frac{d\pi_j(t)}{dt} = q_{jj}\pi_j(t) + \sum_{i \neq j} q_{ij}\pi_i(t)$$

exactly the same as (**)

Therefore the state transition rate diagram contains the exact same information as the transition rate matrix **Q**

HOMOGENEOUS CTMC
State Probabilities

Steady-State Analysis

Q.: What is the probability of finding a MC at state *i* in the long run, i.e., after a period of time long enough so that the state probabilities have reached fixed values which do not change with time?

$$\pi_j = \lim_{t \to \infty} \pi_j(t)$$

Issues to be addressed:

- under what conditions do the above limits exist?
- if they exist, do they form a probability distribution, i.e., $\sum_{\text{all } j} \pi_j = 1$?
- how do we evaluate π_j ?

If π_j exists for some state *j*, it is referred as the *steady-state*, *equilibrium* or *stationary state probability*. If this is true for all states *j*, we obtain the *stationary probability vector* $\pi = [\pi_0, \pi_1, ..]$

HOMOGENEOUS CTMC
State Probabilities

Steady-State Analysis

If the limits exist $\dfrac{d\pi(t)}{dt} = 0 \Rightarrow \pi\mathbf{Q} = \mathbf{0}$

All the results for CTMC parallel those for DTMC. We will state only the most relevant result.

Theorem 11: In an irreducible CTMC consisting of *positive recurrent* states, a unique stationary state probability vector π exists such that $\pi_j > 0$ and

$$\pi_j = \lim_{t \to \infty} \pi_j(t)$$

and the steady state probabilities are determined by solving

$$\sum_{\text{all } j} \pi_j = 1$$

$$\pi\mathrm{Q} = 0$$

RELATION WITH ETPNs

State space of the equivalent CTMC: reachability set $R[x_0]$ of the exponential timed Petri net

Computation of the transition rate from state x_i to state $x_j \neq x_i$ is given by

$$q_{ij} = \sum_{t_k \in T_{ij}} \lambda_k(x_i)$$

Where T_{ij} is the subset of T_D enabled transitions in x_i such that the firing of any transition in T_{ij} leaves the CTMC in x_j .

If $x_j = x_i$,

$$q_{ii} = -\sum_{j \neq i} q_{ij}$$

MARKOV CHAINS

Further reading

• Birth-Death chains – *special structure facilitates the task of obtaining explicit solutions for state probabilities (steady-state and transient analysis*).

• Lots of literature on Markov Chains

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